**[6.62] Try asymptotic expansion for functions of large arguments**

This tip shows a function to generate an asymptotic series expansion for a function. As the name implies, the asymptotic series is useful for evaluating functions at large arguments. The present example shows that asymptotic expansion can fix a severe round-off error problem.

In June 2002, a post on the TI-89 / TI-92 Plus discussion group reported problems evaluating this function for large arguments:

$$f(x) = 1 - x + \frac{1}{2}\left(x - \frac{1}{x} + \frac{4}{6x + 3(x-1)(x+1)\cdot\ln\left(\frac{2x}{x+1} - 1\right)} \right) \qquad [1]$$

This function resulted from a tidal force analysis. Plotting the function from x = 100 to x = 1000 confirms there is definitely a problem:
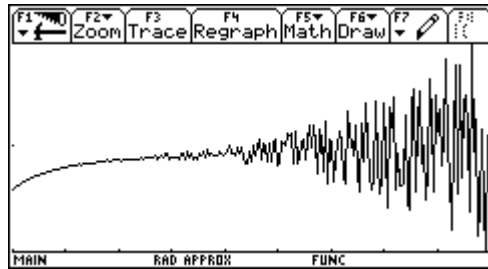
The function starts getting erratic at about 370, and really gets chaotic at about x = 530. Since the limit of f(x) is one, there are definite problems in evaluating the function. As the problem occurs for large *x*, an asymptotic expansion is a potential solution. An asymptotic expansion is a series of the form

$$a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \ldots \qquad [2]$$

which is defined for sufficiently large x, if, for every n = 0, 1, 2, 3, ...

$$\left[f(x) - \left(a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots \right) \right] x^n \to 0 \qquad \text{as} \qquad x \to \infty \qquad [3]$$

then

$$f(x) \approx a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots \qquad [4]$$

Note that [2] may not converge for any *x*. In general, finding an asymptotic expansion is difficult, and many books have been written about the difficulties and some solutions. However, since we are only interested in a numeric approximation, our task is a little easier. Since [3] is defined for every value of n = 1, 2, 3, ..., we find a_0 by letting *n* = 0, then [3] becomes

$$f(x) - a_0 \to 0 \qquad \text{or} \qquad a_0 = \lim_{x\to\infty} f(x)$$

To find the second term a_1, we have

$$\left[f(x) - a_0 - \frac{a_1}{x} \right] x \to 0 \qquad \text{or} \qquad a1 = \lim_{x \to \infty} \left[f(x) - a_0 \right] x$$

We can find as many terms as necessary by repeating this process, which is automated by this function *asympexp()*:

```
asympexp(fx,x,n)
Func
©(f(x),x,number of terms) Asymptotic expansion
©returns {an,...,aØ}
©4junØ2/dburkett@infinet.com

local i,j,a

{}→a

limit(fx,x,∞)→a[1]

for i,1,n-1
 limit(x^i*(fx-Σ(a[j+1]/x^j,j,Ø,i-1)),x,∞)→a[i+1]
endfor

seq(a[i],i,dim(a),1,¯1)

EndFunc
```

asympexp() should be run in Exact mode, since it used the *Limit()* function. *asympexp()* runs very slowly, but eventually returns these terms:

$a_0 = 1$ $\qquad a_1 = -\frac{3}{5}$ $\qquad a_2 = 0$ $\qquad a_3 = -\frac{4}{175}$

$a_4 = 0$ $\qquad a5 = -\frac{4}{375}$ $\qquad a_6 = 0$ $\qquad a_7 = -\frac{6364}{1010625}$

$a_8 = 0$ $\qquad a_9 = -\frac{275708}{65690625}$ $\qquad a_{10} = 0$ $\qquad a_{11} = -\frac{20793692}{6897515625}$

$a_{12} = 0$ $\qquad a_{13} = -\frac{-1335915116}{596288828125}$ $\qquad a_{14} = 0$

$a_{15} = -\frac{7665058771652}{4288702777734375}$ $\qquad a_{16} = 0$ $\qquad a_{17} = -\frac{92763867329564}{643330541666015625}$

and approximate values for the non-zero terms are

$a_0 = 1$ $\qquad\qquad a_9 = $ -4.1970 6769 4210 6 E-3

$a_1 = $ -0.6 $\qquad\qquad a_{11} = $ -3.0146 6399 3602 8 E-3

$a_3 = $ -2.2857 1428 5714 3 E-2 $\qquad a_{13} = $ -2.2785 9555 2080 3 E-3

$a_5 = $ -1.0666 6666 6666 7 E-2 $\qquad a_{15} = $ -1.7872 6742 5349 8 E-3

$a_7 = $ -6.2970 9338 2807 7 E-3 $\qquad a_{17} = $ -1.4419 8797 2232 0 E-3

Note that *asympexp()* returns the coefficients as a list in the form $\{a_n, a_{n-1}, ..., a_0\}$, so the expansion can be evaluated as

```
polyEval(a,1/x)
```

where *a* is the list of coefficients. In general, you would keep finding additional terms until they started to increase in value, then the best approximation would use all the still-decreasing terms. Even though the terms above are still decreasing, I decided to check the expansion with the terms through a_9. This plot shows the original function as points, the asymptotic expansion as a solid line, on the interval x = [100, 1000].
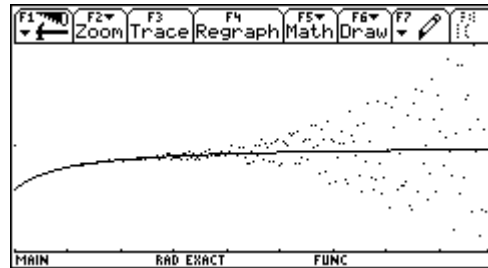
This is obviously a considerable improvement over the original function.

If we want to find both large and small values of the original function, we need to know the value of *x* at which to switch to the asymptotic function. One starting point is to plot the difference of the two functions, and switch to the asymptotic function before the round-off noise the original function begins. I could just plot the difference of the two functions, but the dynamic range is so large that the round-off noise would not be seen, so I plot the natural logarithm of the absolute value of the difference between the two functions, which essentially gives me a plot with a logarithmic y-axis. It looks like this:
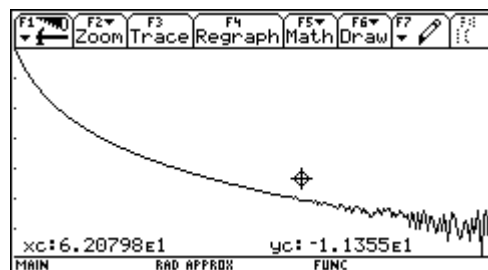
The plot shows the difference between the two function on the interval x = [5, 100]. The cursor readout shows that we could use x = 60 as the transition point between the functions. The function value at this point is about 0.989999885775, and the difference between the functions is about 6.3E-6. If we only need 5 or 6 significant digits in this range, the approximation is good enough. If we need more significant digits, we could increase the number of terms in the asymptotic expansion, but adding four more terms does not decrease the difference between the functions. We would need a lot more terms to increase the accuracy.

For more details on asymptotic expansions, you might start with section 18.6 of *Advanced Engineering Mathematics*, 6th edition, by Erwin Krezszig, Wiley, 1988. The entry for *asymptotic series* at Eric Weisstein's World of Mathematics site (*http://mathworld.wolfram.com/AsymptoticSeries.html*) also has a good bibliography.